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The Directional Profit Efficiency Measure: On Why Profit Inefficiency is either Technical or Allocative

José L. Zofío[a,*], Jesús T. Pastor[b] , Juan Aparicio[b]

[(a)] *Departamento de Análisis Económico*: *Teoría Económica e Historia Económica. Universidad Autónoma de Madrid, E-28049 Madrid, Spain.*

[(b)] *Center of Operations Research* (*CIO*)
Universidad Miguel Hernández, E-03202 Elche, Alicante, Spain

Abstract

The directional distance function has been introduced in the efficiency literature with the intention of relaxing the fixed orientations represented by its classical input and output counterparts. However, the criteria underlying the choice of its associated directional vector are numerous. When market prices are observed and firms have a profit maximizing behavior, it seems natural to choose as directional vector that projecting inefficient firms towards profit maximizing benchmarks. Based on that choice of directional vector, we introduce the profit efficiency measure and show that, in this general setting, profit inefficiency can be categorized as either technical—for firms situating in the interior of the technology—or allocative—for firms lying on the frontier. We implement and illustrate the analytical model by way of Data Envelopment Analysis techniques, where the profit maximizing benchmark may not be unique, and introduce the necessary optimizing program for profit inefficiency measurement.

Key Words: Directional Distance Function, Profit Efficiency, Technical Efficiency, Allocative Efficiency, Data Envelopment Analysis.

JEL Classification: C61, D21, D24.

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1. Introduction

In the last decade several methodologies have been proposed to measure firm's profit inefficiency and decompose it into meaningful and mutually exclusive terms informing about technical and allocative performance. The existing literature differentiates between the radial or equiproportional oriented approach developed by economists relying on the rationality of duality theory, and non-oriented approaches proposed from the field of operations research and management science, mainly concerned with the practicality of the analysis. Among the former we find the proposals by Banker and Maindiratta (1988), Chambers et al. (1998), Chavas and Cox (1999) and Asmild et al. (2007). Among the latter we find several approaches based on the additive model introduced by Charnes et al. (1985), including Cooper et al. (1999), Portela and Thanassoulis (2005, 2007) and Ruiz and Sirvent (2010). These last authors discuss the particularities of each of these contributions, whose common goal is to inform managers on ways to change their relative demand for inputs and supply of outputs at the existing market prices, so as to increase their profits.

In the process of increasing observed profit efficiency with respect to maximum profit, it is assumed that managers may adopt better intra firm organizational practices reducing technical inefficiency—a first option that increases observed profit since reaching the production frontier implies the contraction of inputs and/or expansion of outputs, as well as changing the demand for inputs and/or the relative supply of outputs—a second option also increasing observed profit since these changes in input and output mixes reduces allocative inefficiency. In short, from a *production analysis—black box— perspective* technical inefficiency can be though of as wrong engineering practices since it only depends on quantities, while allocative inefficiency is due to economic mismanagement since firms do not demand and/or supply the amounts of inputs and outputs that maximize profit at their market prices. With regard to the reduction in technical inefficiency both the economic and OR/MS approaches impose that inputs and outputs must be physically reduced and/or increased, respectively, while when dealing with allocative inefficiency, the change in input and output mixes does not impose any restrictions on the inputs and outputs adjustments, i.e. some—or all— inputs can be increased while some—or all—outputs can be reduced. In this paper we argue that this restriction on inputs and outputs adjustments when dealing with technical inefficiency is related to the initial definition of the partially and radially oriented input and output distance functions upon which efficiency measurement and its decomposition is based. A restriction that is compatible with the notion of Pareto-Koopmans efficiency normally considered in the OR/MS field.

The main feature of our proposal is that, contrary to the previous approaches, we make the choice of the orientation leading to profit efficiency endogenous, and relax the restrictions imposed on inputs and outputs variations when reducing technical inefficiency. By allowing any change in their relative values—as long as profit efficiency increases, our methodology represents a flexible framework for profit efficiency measurement; [1] whose main outcome is the categorization of profit inefficiency as technical or allocative, but not both. From a *theoretical—conceptual—perspective* we base this result on the fact that the conventional decomposition of overall profit efficiency into technical and allocative terms have a clear interpretation for homothetic technologies; a quite strong theoretical assumption that is normally overlooked by applied researchers and that we do not impose in our analysis. This result is quite significant since many efforts have been devoted to the conceptual, theoretical, and applied study of the decomposition of productive efficiency. For homothetic technologies the marginal rate of substitution/transformation along a given ray vector remains constant—isoclines are ray lines or factor beams, and therefore the traditional input reducing or output increasing distance functions do not change the allocative efficiency of an inefficient unit when projected to its reference benchmark on the frontier. As a result the level of allocative (in)efficiency also remains constant when reducing technical inefficiency radially. But for non homothetic technologies the allocative efficiency of firms lying inside the production possibility set cannot be ascertained unless one is willing to assume that the relevant information on the marginal rate of substitution/transformation may be recovered from the technology (i.e., assuming a reference isoquant—different from the technical efficiency frontier—upon which marginal rates can be determined). Bogetoft et al. (2006) were the first authors to underline this analytical difficulty when decomposing overall efficiency into its technical and allocative components, and the fact that the latter cannot be unambiguously established. Allowing for both homothetic and non-homothetic technologies, they propose alternative ways to decompose cost inefficiency differentiating between the standard and "reverse" Farrell approaches. They conclude that the values associated to technical and allocative efficiencies are order dependent unless a "strong consistency requirement" (the technology being homothetic) is satisfied. Since this difficulty arises from the fact that firms are technically inefficient and we do not introduce additional information to determine the allocative efficiency of firms lying inside the production possibility set, we attribute overall profit inefficiency to technical reasons.

[1] Ray (2007) introduced a measure of (shadow) profit inefficiency through the normalization of the shadow cost of the actual input bundle and the shadow revenue of the actual output bundle. This approach has several implications in the quantity space. In particular, it allows any change in the values of inputs and outputs of the assessed firm. In other words, it is possible that some–or all–inputs can be increased while some–or all–outputs can be reduced in order to reach the frontier.

From a *managerial and organizational perspective* this result is critical. If we take the results of the theoretical model literally to prescribe increasing profit actions to managers —and not as a way to conceptually characterize and decompose profit inefficiency, current two-step analysis may advice them to undertake a technically efficiency improving strategy that implies inputs reduction and outputs increases on a first instance —reaching the production frontier (first step), and later on engage in changes in the input and output mixes that may increase the former and reduce the latter so as to increase allocative efficiency (second step). This is a contradiction that should not be generally accepted in theoretical or empirical analyses unless a convincing justification exists. The advantage of our proposal is that the directions given to managers with respect to input and outputs adjustments are univocal, thereby preventing a conflict between technical and allocative improvement strategies. In a sense, what we are stating is that organizational change processes taken place in reality will not normally follow prescriptions that are constrained by theoretical results pervading empirical modeling. On the contrary, Bogetoft et al (2006) argue that in some situations it may be favorable to undertake non-monotone adjustments of inputs and outputs–e.g. some resources are first laid off and then reemployed back again (or vice versa), particularly in the case of human resources and based on motivational grounds, thereby justifying the need for the two-step procedure emanated from the theoretical model. In reality technical and allocative inefficiencies are dealt with simultaneously by managers. Therefore, even if the conceptual distinction between technical inefficiency—as an *organizational slack* or *X-Ineffiency*, see Cyert and March's (1963) and Liebenstein (1966), respectively—and allocative inefficiency—as a market or behavioral inefficiency, may be relevant to plan improving strategies, it seem reasonable to advise managers about the necessary organizational changes that they have to undertake to attain overall efficiency, which can then be categorized as technical or allocative.

In this article we embed our proposal within the economic approach characterized by an oriented profit efficiency measurement. Specifically we adopt the additive framework of the shortage function proposed by Luenberger (1992), as interpreted by Chambers et. al. (1996, 1998) trough their directional distance function, DDF, constituting the technical efficiency component within a decomposable profit efficiency measure. As opposed to its traditional input and output counterparts, the DDF implies a directional vector thereby allowing for a flexible choice of orientation in the inputs and outputs dimensions. We anticipate that since the particular values of the directional vector are exogenously established by researchers, the existing decomposition of profit efficiency into its technical and allocative terms depends upon that subjective choice. Finally, even if we adopt the analytical structure represented by the DDF to introduce the directional profit efficiency measure, we stress the fact that the same analytical framework for profit efficiency measurement could be developed for alternative oriented models—e.g. the generalized distance function of Chavas and Cox (1999), or non oriented models—e.g. Portela and Thanassoulis's (2007) geometric distance function. This means that the ideas presented in this paper can be extended to all proposals dealing with profit efficiency measurement.

The article proceeds as follows. In the next section we present the existing decomposable measure of profit efficiency based on the directional distance function, which can be regarded as a measure of technical efficiency, and where the restricted directional vector reducing inputs and increasing outputs is exogenously chosen. Here we propose a choice of the directional vector that takes into account market prices resulting in a value that measures economic inefficiency in currency units, i.e. in terms of foregone profit due to technical inefficiency. In section 3 we introduce the directional profit efficiency measure, thereby endogenizing the directional vector and lifting any restriction on inputs and outputs adjustments. This measure yields overall foregone profit due to technical or allocative inefficiencies and, resorting to the previous directional distance function, we show that profit inefficiency is either technical or allocative. In section 4, we render our profit efficiency measure operational within a Data Envelopment Analysis framework and introduce the optimizing programs for its calculation. In section 5 we exemplify the model using a simple data set. Since in this framework the profit maximizing benchmark may not be unique, we propose a quadratic program that calculates the minimum distance to the closest benchmark along the optimal directional vector. Section 6 concludes with the practical managerial implications of the new approach in terms of profit improving strategies.

2. The existing profit efficiency decomposition based on the directional distance function

In this section we present the technology, the definition of the directional distance function proposed by Chambers et al (1996, 1998), and introduce the directional profit inefficiency measure. Let us consider a panel of $i = 1,...,I$ firms transforming input vectors $x_i = (x_{1i},..., x_{Ni}) \in R_+^N$ into output vectors $y_i = (y_{1i},..., y_{Mi}) \in R_+^M$. The technology can be represented by the production possibility set:

$$T = \{(x, y): x \text{ can produce } y\}, \tag{1}$$

and we assume the standard axioms discussed in Färe and Primont (1995)—particularly convexity, closeness and outputs and inputs free disposability.

The directional distance function defines in terms of T as the maximum feasible expansion of the output vector and reduction of the input vector:

$$D_T\left(x,y;g_x,g_y\right)=\max_\beta\left\{\beta:(x-\beta g_x,y+\beta g_x)\in T\right\},\quad x\in R_+^N,y\in R_+^M. \tag{2}$$

The directional function (2) expands outputs and contracts inputs by adding and subtracting the amount represented by β times the elements of the preassigned non-zero directional vector g = $\left(g_x,g_y\right)\in R_+^N\times R_+^M\setminus\left\{0_{N+M}\right\}$. Actually, since βg_x is substracted from x, the direction $\left(g_x,g_y\right)$ is graphically represented by $\left(-g_x,g_y\right)$, *forcing* the reduction of inputs and the increment of outputs. The directional vector represents the relative weights that the directional distance function places on outputs and inputs when moving toward the best practice production frontier represented by the boundary of the technology. Chambers *et al* (1998) discuss its properties, including translation, continuity, monotonicity and concavity. The directional distance function can be interpreted as a measure of technical efficiency in the sense of Farrell (1957) for any vector $(x,y)\in T$ by measuring its distance to the best practice frontier represented by the boundary of the technology in the preassigned direction $\left(g_x,g_y\right)$.

We now recall that when choosing the profit function as the benchmark against which to confront economic performance, the difference between maximum profit and the value of the directional distance function can be interpreted, through Malher's inequality, as a measure of allocative inefficiency. Denoting the output and input market price vectors by $p\in R_{++}^M$ and $w\in R_{++}^N$ the profit function defines as:

$$\pi(p,w)=\max_{x,y}\{py-wx:(x,y)\in T\}, \tag{3}$$

From the definition of the profit function we observe that $\pi(p,w)$ is greater than or equal to the observed value attained at any feasible input–output vector, *i.e.*

$$\pi(p,w)\geq py-wx,\forall\,(x,y)\in T. \tag{4}$$

For any input–output vector belonging to the production technology (1), the projected vector $(x-D_T(x,y;g_x,g_y)\,g_x,\ y+D_T(x,y;g_x,g_y)\,g_y)\in T$, i.e. is feasible, and we observe that:

$$\pi(p,w)\geq p(y+D_T(x,y,g_x,g_y)g_y)-w(x-D_T(x,y,g_x,g_y)g_x), \tag{5}$$

This formulation showing the relationship between the profit and the directional distance function corresponds to an additive Mahler inequality, Färe and Grosskopf (2000a), i.e.

$$D_T(x,y;g_x,g_y) \leq \frac{\pi\left(p,w\right)-(py-wx)}{\left(pg_y+wg_x\right)} \quad . \tag{6}$$

From this expression it is possible to decompose profit inefficiency. Let us consider its right hand side as a measure of overall profit efficiency comparing observed profit to maximum profit: OPE = $[\pi\ (p, w) - (\ py - wx\ )]\ /\ (pg_y + wg_x)$, [2] and recall the technical efficiency interpretation of the directional distance function: TE $= D_T(x,y;g_x,g_y)$. Then (6) can be rendered an equality by introducing a residual allocative efficiency term: AE. This term measures inefficiency due to failure to choose the profit maximizing input–output vector at the market prices when projected to the production frontier in the $\left(g_x, g_y\right)$ orientation, *i.e.* the distance between profit at the technically efficient projection and maximum profit. Closing the inequality in (6) yields:

$$\mathrm{AE}(x,y,p,w;g_x,g_y) = \frac{\pi(p,w)-(py-wx)}{pg_y+wg_x} - D_T(x,y;g_x,g_y)\,, \tag{7}$$

which can be expressed in the standard Farrell's (1957) type decomposition of profit efficiency, but additively: OPE = TE + AE .

With regard to some key values of OPE, it is nonnegative for any feasible production firm (*x*,*y*). If observed profit equals maximum profit (6) is verified as an equality with TE = AE = 0 and the firm is profit efficient; if observed profit is smaller than maximum profit the firm is profit inefficient and we have two situations: i) when the difference is equal to $D_T(x,y;g_x,g_y)$ all inefficiency is technical, ii) but if (6) is an inequality, the difference (residual) between both values corresponds to allocative inefficiency.

We remark that in their theoretical papers, Chambers et al (1996, 1998) do not specify a particular orientation $\left(g_x, g_y\right)$ to introduce their decomposable measure of profit efficiency, but we can trivially see from (6) and (7) that choosing an orientation such that $\left(pg_y + wg_x\right)$ = 1 will

[2] The overall profit efficiency corresponds to the concept of Nerlovian efficiency presented by Chambers *et al.* (1998).

result in monetary values of overall, technical and allocative efficiencies. Under this restriction, the directional distance function $D_T(x, y; g_x, g_y)$ equals foregone or unrealized profit due to technical inefficiency. Let us denote by $g^T = \left(g_x^T, g_y^T\right) \in R_{++}^N \times R_{++}^M$ a particular orientation projecting firm (x, y) onto the frontier vector $\left(x^T, y^T\right)$ while satisfying $\left(pg_y^T + wg_x^T\right) = 1$. In this way, the directional vector can be defined as $\left(g_x^T, g_y^T\right) = \varsigma\left(x - x^T, y^T - y\right)$, where the parameter $\varsigma > 0$ is a scalar related to the value of the directional distance function in the following way: $D_T\left(x, y; g_x^T, g_y^T\right) = 1/\varsigma$.

Proposition 1. Let (p, w) be the vector of market prices. Let $(x, y) \in T$, then $D_T\left(x, y; g_x^T, g_y^T\right) = 1/\varsigma$.

Proof.

As previously shown, for any input–output vector belonging to the production technology (1), the projected vector $\left(x^T, y^T\right)$ = $(x - D_T(x, y; g_x^T, g_y^T)\ g_x^T,\ y + D_T(x, y; g_x^T, g_y^T)\ g_y^T) \in T$, i.e. is feasible. As a result we have:

$$\begin{aligned}
py^T - wx^T &= p(y + D_T(x, y; g_x^T, g_y^T)g_y^T) - w(x - D_T(x, y; g_x^T, g_y^T)g_x^T) \\
&= p\left(y + \beta^* g_y^T\right) - w\left(x - \beta^* g_x^T\right) \text{ [using } \beta^* \text{ as the solution to (2)]} \\
&= py - wx + \beta^*(pg_y^T + wg_x^T) \\
&= (py - wx) + \beta^*\left(p\varsigma\left(y^T - y\right) + w\varsigma\left(x - x^T\right)\right) \text{[substituting } \left(g_x^T, g_y^T\right) = \varsigma\left(x - x^T, y^T - y\right)\text{]} \\
&= (py - wx) + \beta^*\varsigma\left(p\left(y^T - y\right) + w\left(x - x^T\right)\right),
\end{aligned}$$

and rearranging we obtain:

$$\left((py^T - wx^T) - (py - wx)\right)/\varsigma = \beta^*\left(p\left(y^T - y\right) + w\left(x - x^T\right)\right).$$

Since $py^T - wx^T - (py - wx) = \left(p\left(y^T - y\right) + w\left(x - x^T\right)\right)$, then $1/\varsigma = D_T\left(x, y; g_x^T, g_y^T\right)$.

■

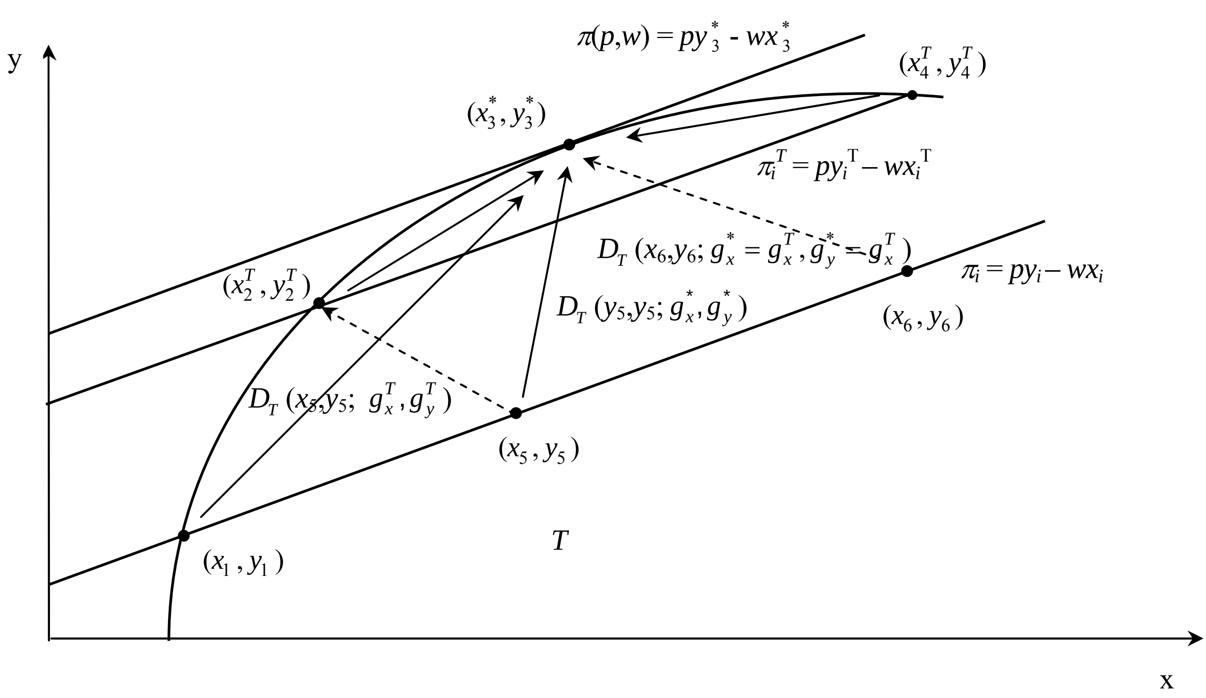
We remark that $D_T\left(x, y; g_x^T, g_y^T\right)$ depends on the particular choice of directional vector $\left(g_x^T, g_y^T\right)$ as long as it satisfies the price normalization constraint—i.e. the directional vector is not unique, as a result the parameter ς also depends on that choice through the projected vector $\left(x^T, y^T\right)$. But regardless of that choice, $\varsigma := \left[\left(py^T - wx^T\right) - \left(py - wx\right)\right]^{-1}$, implying that

$D_T\left(x,y;g_x^T,g_y^T\right) = \left(py^T - wx^T\right) - \left(py - wx\right)$ is a natural measure of technical efficiency in monetary terms.

Figure 1 illustrates the process of profit efficiency measurement and its decomposition based on the directional distance function. Given the vector of prices (p,w), firm (x_3^*, y_3^*) maximizes profit: $\pi(p,w) = py_3^* - wx_3^*$, representing the economic benchmark for all the remaining inefficient firms. Initially exploring the case of technically inefficient firms lying inside the production possibility set, e.g. (x_5, y_5), and choosing as directional vector one that satisfies the normalizing restriction $pg_y + wg_x = 1$, so $(g_x, g_y) = (g_x^T, g_y^T)$, the chosen directional distance function projects (x_5, y_5) precisely onto the second firm (x_2^T, y_2^T) resulting in a profit (allocative) inefficient benchmark vector. As a result profit inefficiency is due to both technical and allocative reasons: TE = $D_T(x_5, y_5;\ g_x^T,\ g_y^T) = (py_2^T - wx_2^T) - (py_5 - wx_5) = \pi_2^T - \pi_5 > 0$ and AE = $\pi(p,w) - (py_2^T - wx_2^T) = \pi(p,w) - \pi_2^T > 0$, with OPE = TE + AE = $\pi(p,w) - (py_5 - wx_5) = \pi(p,w) - \pi_5 > 0$. This case allows us to discuss the managerial or organizational issue regarding the two-step non monotone procedure of adjusting inputs and outputs so as to reach profit efficiency. The choice of a directional vector reducing inputs and/or increasing outputs $(g_x, g_y) = (g_x^T, g_y^T)$ results in a movement that increases profit efficiency by eliminating technical efficiency; but its projection (x_2^T, y_2^T) on the frontier is allocative inefficient, so further adjustments are needed to gain allocative efficiency and, thereby, profit efficiency. However, this second step requires an increase in inputs, suggesting that the previous reduction represented a contradictory organizational change due to the subjective choice of the directional vector. Note that choosing the directional vector according to this or similar criterion (including that corresponding to the observed input—output vector) will result in a two-step procedure with profit inefficiency being both technical and allocative, and leading to conflicting strategies.[3] The only exception is represented by firm (x_6, y_6), whose projection in the (g_x^T, g_y^T) direction by way of $D_T(x_6, y_6;\ g_x^T,\ g_y^T) > 0$ happens to be precisely the profit maximizing benchmark (x_3^*, y_3^*). In this particular case, while the directional vector reduces the input and increases the output as assumed by (2), all profit inefficiency is technical. Finally, for the technically efficient firms (x_1^T, y_1^T), (x_2^T, y_2^T) and (x_4^T, y_4^T), all profit inefficiency is allocative, and regardless the choice of orientation (g_x^T, g_y^T) satisfying the normalizing constraint: $D_T(x_1, y_1;\ g_x^T,\ g_y^T) = D_T(x_2, y_2;\ g_x^T,\ g_y^T) = D_T(x_4, y_4;\ g_x^T,\ g_y^T) = 0$.

[3] Färe and Grosskopf (2000b:98) propose that the direction should be given by the observed input—output bundle of the firm, $(-g_x, g_y) = (-x_5, y_5)$: "Our rationale is that this provides a link and symmetry with the

Figure 1. Profit efficiency measurement



3 The directional profit efficiency measure and the characterization of profit inefficiency

Assuming that the final goal of the managers is to attain maximum profit, the most sensible choice for the directional vector in case of profit inefficiency is that projecting any firm to the profit maximizing benchmark. From a theoretical perspective it requires a flexible modelling approach unconstrained by an exogenously chosen direction and the existing restrictions relative to inputs reductions and outputs increases, which may result in conflicting efficiency improving strategies from a managerial perspective. Therefore, as opposed to the standard directional distance function model, this means that the directional vector must be rendered endogenous, since it is the result of a comparative process that takes into account the economic benchmark, which in turn requires lifting the restrictions on inputs and outputs adjustments. In this section we show that by endogenizing the choice of direction we can define a measure of profit inefficiency that disposes of the allocative residual emanating from the Mahler's inequality presented in (6), and along with the directional distance function, allows the characterization of eventual inefficiencies as either technical or allocative.

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traditional distance functions which are defined in the direction of the observed input and output mix for each observation". Countless applied research has been performed taking into account this consideration.

To this end, we now select a specific reference directional vector $\left(g_x^*, g_y^*\right) \in R^N \times R^M \setminus \{0_{N+M}\}$ which allows us to project firm (x, y) onto the maximizing profit input–output vector $\left(x^*, y^*\right)$. The directional vector is defined as $\left(g_x^*, g_y^*\right) = \tau\left(x - x^*, y^* - y\right)$, where the parameter $\tau > 0$ is a scalar, whose value also relates to the length of the directional profit efficiency measure as we show next.

To define the directional profit inefficiency measure associated to the directional distance function we use the optimal reference directional vector $\left(g_x^*, g_y^*\right)$ for any $(x, y) \in T$, and assume that maximum profit is not achieved at this point, i.e. (x, y) is inefficient: [4]

$$D_T\left(x, y; g_x^*, g_y^*\right) = \max_\beta \left\{\beta : \left(x - \beta g_x^*, y + \beta g_y^*\right) \in T\right\}, \tag{8}$$

which coincides with the definition of the directional distance function (2) with a significant variation. In (8) we chose a maximizing profit directional vector $\left(g_x^*, g_y^*\right)$ that may have negative components, *i.e.*, it is possible to increase inputs and to decrease outputs through $\left(g_x^*, g_y^*\right)$ to reach the frontier and no restrictions are imposed on the directional vector, as opposed to the directional distance function (2), where the arbitrary reference vector $\left(g_x, g_y\right)$ always has nonnegative components resulting in the general $\left(-g_x, g_y\right)$ direction, *i.e.*, inputs are reduced and outputs are increased to reach the frontier—including the particular cases of directions complying with the price normalization restriction, $\left(g_x^T, g_y^T\right)$, and the one used in most empirical studies $\left(g_x, g_y\right) = (x, y)$

Lemma 1. Let (p, w) be the vector of market prices. Let $(x, y) \in T$ such that $\pi(p, w) > py - wx$. Then, $D_T\left(x, y; g_x^*, g_y^*\right) = 1/\tau$.

Proof. On the one hand, $\beta = 1/\tau$ is a feasible solution of (8) since $\left(x - \frac{1}{\tau}\tau\left(x - x^*\right), y + \frac{1}{\tau}\tau\left(y^* - y\right)\right) = \left(x^*, y^*\right) \in T$. On the other hand, let us assume that $\hat{\beta} > 1/\tau$ is a feasible solution as well. Then, $\left(x - \hat{\beta} g_x^*, y + \hat{\beta} g_y^*\right) \in T$. So,

$$p\left(y + \hat{\beta} g_y^*\right) - w\left(x - \hat{\beta} g_x^*\right) = (py - wx) + \hat{\beta}\left(p g_y^* + w g_x^*\right) =$$

[4] Otherwhise $g^* = 0_{N+M}$, which would lead to a not well defined optimization model for $D_T\left(x, y; g_x^*, g_y^*\right)$.

$$= (py - wx) + \hat{\beta}\tau\left(p\left(y^* - y\right) + w\left(x - x^*\right)\right) > \left(py^* - wx^*\right) = \pi(p,w).$$

The last inequality leads to a contradiction. Consequently, $D_T\left(x, y; g_x^*, g_y^*\right) = 1/\tau$. ■

As with the standard directional distance function (2), $D_T\left(x, y; g_x^*, g_y^*\right)$ can be derived from the profit function as well.

Proposition 2. Let (p,w) be the vector of market prices. Let $(x,y) \in T$ such that $\pi(p,w) > py - wx$. Then,

$$D_T\left(x, y; g_x^*, g_y^*\right) = \min_{p,w}\left\{\pi(p,w) - (py - wx) : pg_y^* + wg_x^* = 1\right\}. \tag{9}$$

Proof. First, we observe that the vector $\dfrac{(p,w)}{pg_y^* + wg_x^*} = \dfrac{(p,w)}{\tau\left(\pi(p,w) - (py - wx)\right)} > 0_{N+M}$ satisfies the constraint $pg_y^* + wg_x^* = 1$, and it is easy to check that the value of the objective function at this vector is equal to $1/\tau$. So, we have that $\min_{p,w}\left\{\pi(p,w) - (py - wx) : pg_y^* + w_x^* = 1\right\} \leq 1/\tau = D_T\left(x, y; g_x^*, g_y^*\right)$, where the last equality holds thanks to Lemma 1. To prove the reverse inequality, we note that as in the directional distance function case $\left(x - D_T\left(x, y; g_x^*, g_y^*\right)g_x^*, y + D_T\left(x, y; g_x^*, g_y^*\right)g_y^*\right) \in T$. Then, by the definition of the profit function, we have that for all (p,w) such that $pg_y^* + wg_x^* = 1$,

$$\pi(p,w) \geq p\left(y + D_T\left(x, y; g_x^*, g_y^*\right)g_y^*\right) - w\left(x - D_T\left(x, y; g_x^*, g_y^*\right)g_x^*\right) =$$
$$= py - wx + D_T\left(x, y; g_x^*, g_y^*\right)\left(pg_y^* + wg_x^*\right) = py - wx + D_T\left(x, y; g_x^*, g_y^*\right).$$

Rearranging terms, we obtain that $D_T\left(x, y; g_x^*, g_y^*\right) \leq \pi(p,w) - (py - wx)$. And, by definition of minimum, we finally have that $D_T\left(x, y; g_x^*, g_y^*\right) \leq \min_{p,w}\left\{\pi(p,w) - (py - wx) : pg_y^* + w_x^* = 1\right\}$. ■

Also, using Proposition 2 and since $\dfrac{(p,w)}{pg_y^* + wg_x^*}$ satisfies the constraint $pg_y^* + wg_x^* = 1$, we obtain the following Mahler inequality mirroring (6):

$$D_T\left(x,y;g_x^*,g_y^*\right)\leq\frac{\pi\left(p,w\right)-\left(py-wx\right)}{pg_y^*+wg_x^*}. \tag{10}$$

Next, we show that in the last expression the equality always holds. In other words, the new distance (8) can be interpreted as a measure of overall profit efficiency instead of technical efficiency as usual. From Lemma 1, we know that $\left(x-D_T\left(x,y;g_x^*,g_y^*\right)g_x^*,y+D_T\left(x,y;g_x^*,g_y^*\right)g_y^*\right)$ = $\left(x-\frac{1}{\tau}\tau\left(x-x^*\right),y+\frac{1}{\tau}\tau\left(y^*-y\right)\right)=\left(x^*,y^*\right)$. Then, if we calculate profit at market prices at this point, we have:

$$\begin{aligned}\pi\left(p,w\right)=py^*-wx^*&=p\left(y+D_T\left(x^*,y^*;g_x^*,g_y^*\right)g_y^*\right)-w\left(x-D_T\left(x^*,y^*;g_x^*,g_y^*\right)g_x^*\right)=\\&=py-wx+D_T\left(x^*,y^*;g_x^*,g_y^*\right)\left(pg_y^*+wg_x^*\right).\end{aligned} \tag{11}$$

Finally, rearranging terms, we achieve the desired equality, i.e.,

$$D_T\left(x,y;g_x^*,g_y^*\right)=\frac{\pi\left(p,w\right)-\left(py-wx\right)}{pg_y^*+wg_x^*}, \tag{12}$$

which implies that the residual identified with allocative efficiency in the directional distance function case fades away when the directional vector leads to the profit maximizing benchmark, i.e., OPE = $D_T\left(x,y;g_x^*,g_y^*\right)$.

It is worth noticing once again that the value of $D_T\left(x,y;g_x^*,g_y^*\right)$ depends on the parameter τ. As in the previous section where $\varsigma:=\left[\left(py^T-wx^T\right)-\left(py-wx\right)\right]^{-1}$, we now consider $\tau:=\left[\pi\left(p,w\right)-\left(py-wx\right)\right]^{-1}$. It implies that $D_T\left(x,y;g_x^*,g_y^*\right)=\pi\left(p,w\right)-\left(py-wx\right)$ is the natural measure of profit efficiency in monetary terms–currency dollars.

3.1 Profit inefficiency is either technical or allocative.

In terms of the standard Farrell's (1957) type decomposition of profit efficiency, we have shown that the directional profit efficiency measure can be regarded as a measure of overall profit efficiency, OPE = $D_T\left(x, y; g_x^*, g_y^*\right)$. As opposed to its standard directional distance counterpart (2) reflecting technical efficiency, the optimally oriented measure (8) determines profit efficiency defined as the ratio of the difference between maximal and observed profit normalized by the value of the optimal directional vector, (12).

We can use these definitions to conclude that profit inefficiency is either technical or allocative, but not both. On one hand we can show that $D_T\left(x, y; g_x^*, g_y^*\right) \geq D_T\left(x, y; g_x^T, g_y^T\right)$ under the condition that both g^T and g^* satisfy the price normalization restriction, and, based on (7), we would have that $\mathrm{AE}(x, y, p, w; g_x^T, g_y^T) = D_T\left(x, y; g_x^*, g_y^*\right) - D_T\left(x, y; g_x^T, g_y^T\right)$. On the other we know that if $D_T\left(x, y; g_x^T, g_y^T\right) = 0$ it will be technically efficient lying on the production frontier, while if $D_T\left(x, y; g_x^T, g_y^T\right) > 0$ the firm will be technically inefficient situating inside the production possibility set. In the former case: $D_T\left(x, y; g_x^T, g_y^T\right) = 0$, we see that all profit inefficiency would be allocative and, in monetary terms, equal to $D_T\left(x, y; g_x^*, g_y^*\right)$. But in the latter case: $D_T\left(x, y; g_x^T, g_y^T\right) > 0$, we have shown that the directional profit efficiency measure projects the evaluated firm to the profit maximizing benchmark, where it is allocative efficient, i.e. $\mathrm{AE}(x, y, p, w; g_x^T, g_y^T) = 0$ and, therefore, all inefficiency is technical, and equal to $D_T\left(x, y; g_x^*, g_y^*\right)$ in monetary terms. As a result we conclude that overall profit efficiency, directly assessed by way of the directional profit efficiency measure is either technical or allocative.

From a production—black box—analysis perspective the main consequence of the analysis is the characterization of overall profit inefficiency as either technical (wrong engineering practices) or allocative (economic mismanagement when demanding and supplying inputs and outputs). A result that derives from the fact that OPE is obtained by identifying the profit efficiency measure along the optimal direction $\left(g_x^*, g_y^*\right)$ instead of being simply calculated by subtracting observed profit from maximum profit. Accordingly, the endogenous directional vector $\left(g_x^*, g_y^*\right)$ becomes the cornerstone of the overall evaluation of profit efficiency in the orientation that guarantees maximum profit, without relying on intermediate steps forced by an subjective choice of the directional distance vector $\left(g_x, g_y\right)$, e.g., $\left(g_x^T, g_y^T\right)$ satisfying the normalizing constraint, or $\left(x, y\right)$ as in most empirical applications. From a theoretical and conceptual perspective, our proposal solves the arbitrary decomposition of profit efficiency since the relative values of the technical and residual allocative efficiencies depend on the exogenous choice of the directional vector, as it is the case of the standard directional distance function. Finally, from an empirical managerial and organizational perspective, we conclude that if one assumes a profit maximizing behaviour on the part of firms, profit inefficient firms as a result of technical and allocative inefficiency will not be interested normally in intermediate projections towards the production frontier that would bring technical efficiency in the arbitrary direction $\left(g_x, g_y\right)$, but may result later in non-monotone adjustments in inputs and outputs.

We illustrate these issues with a simple single-input, single-output example inspired in Ray (2004: 227). Assume that the production function were $y = f(x) = \alpha x^{\beta}$. Given the output and input prices (p,w), the vector (x^*,y^*) maximizes profit if it satisfies the first order condition: $f'(x) = \alpha\beta / x^{\beta-1}$ = w/p, so the input demand function is $x^* = \left(\alpha\beta / (w/p)\right)^{1/\beta-1}$ and the output supply function is $y^* = \alpha\left(\left(\alpha\beta / (w/p)\right)^{1/\beta-1}\right)^{\beta}$. Assuming initially that $\alpha = 1$ and $\beta = 0.5$ and (p, w) = (2, 1), we see that that the optimal vector of input demand and output supply is (x^*, y^*) = (1, 1), since for $\beta = 0.5$: $y^* = x^* = \alpha^2$, with $\pi(p,w) = p\, y^* - w\, x^* = 1$. Now consider the alternative case where $\alpha = 0.5$ and β and prices remain constant, then (x^*, y^*) = (0.25, 0.25). Therefore, when the production frontier is defined by the initial $\alpha = 1$ production function, the latter vector is technically inefficient lying inside the frontier, but allocative efficient, since for those prices the contraction of the technology represented by the reduction from $\alpha = 1$ to $\alpha = 0.5$ would make it profit efficient. In this case the optimal direction projecting the inefficient vector (0.25, 0.25) to the efficient vector (1, 1) is $\left(g_x^*, g_y^*\right)$ = (0.25–1, 1–0.25) = (–0.75, 0.75), that implies an increase in both the input and the output. Therefore, why would a firm characterized by the production process (0.25, 0.25) move according to a restricted orientation $\left(g_x^T, g_y^T\right)$ imposed by the directional distance function (reducing inputs and increasing outputs) that would reduce technical inefficiency but increase allocative inefficiency at the same time? Moreover, since (0.25, 0.25) satisfies the first order conditions for the α = 0.5 production frontier by being allocative efficient, in this particular example the projection along the optimal direction can be interpreted as technical efficiency gains, and it can be shown that the direction given by the bisecting line $y^*/x^* = 1$ constitutes the expansion path of profit maximizing input–output combinations for alternative α. Firms lying on the production frontier are technically efficient and therefore their projection onto the profit maximizing benchmark solves allocative inefficency, *i.e.*, all profit inefficiency is allocative. Contrarily, for firms lying inside the production frontier, it is impossible to determine their allocative performance unless some plausible technological characteristics at their specific input–output bundle are assumed, *i.e.* as we have illustrated in our particular single input-single output example by determining the marginal productivity of the inefficient firm (0.25, 0.25) by assuming $\alpha = 0.5$ so it belongs to the deflated frontier. However, this proposal based on Bogetoft et al. (2006) departs from the commonly established assumption that the allocative performance of firms can only be established at the production frontier, allowing us to characterize all profit inefficiency as technical.

The relevance of the optimal directional vector (g_y^*, g_x^*) for the measurement of profit efficiency and its characterization can be also graphically illustrated recalling our example in Figure 1. Starting again with the case of technically inefficient firms, e.g. (x_5, y_5), the directional profit efficiency measure projects it onto the profit maximizing benchmark, (x_3^*, y_3^*), with OPE = $\vec{D}_T(x_5, y_5; g_x^*, g_y^*) = \pi(p,w) - \pi_5 > 0$. Since the value of its directional distance function is positive: TE = $\vec{D}_T(x_5, y_5; g_x^T, g_y^T) = (py_2^T - wx_2^T) - (py_5 - wx_5) = \pi_2^T - \pi_5 > 0$, we learn that all profit inefficiency is due to technical reasons. Contrarily to the standard decomposition of profit efficiency that results in a two-step non monotone adjustment of inputs or outputs so as to reach profit efficiency, the directional profit efficiency measure implies a profit enhancing strategy that prevents contradictory moves when demanding inputs and supplying outputs in the market. This translates into clear and non conflicting guidelines to the managers so they can plan consistent and compatible organizational changes leading to the best possible maximizing profit results—this changes may be categorized at will as engineering (technical) and/or market (allocative) oriented. For firm (x_6, y_6) we represent the particular case when the directional profit efficiency measure is equal to the directional distance function: OPE = $\vec{D}_T(x_6, y_6; g_x^*, g_y^*)$ = TE = $\vec{D}_T(x_6, y_6; g_x^T, g_y^T) > 0$, precisely because $(g_x^*, g_y^*) = (g_x^T, g_y^T)$. Again, this situation implies than all profit inefficiency is technical. Finally, for the technically efficient firms: (x_1^T, y_1^T), (x_2^T, y_2^T) and (x_4^T, y_4^T), all profit inefficiency is allocative since their associated directional distance function in the possible (g_x^T, g_y^T) directions are zero; and being equal for latter two firms: OPE = AE = $\vec{D}_T(x_2, y_2; g_x^*, g_y^*) = \vec{D}_T(x_4, y_4; g_x^*, g_y^*) > 0$.

4. Empirical implementation by means of the activity analysis

In this section we show how to implement the search for the optimal directional vector (g_y^*, g_x^*) that ensures the measurement of economic efficiency against maximizing profit by way of the directional profit efficiency measure (8), and determine whether profit inefficiency is due to technical or allocative reasons, which in turn requires calculation of the directional distance function (2) in a (g_y^T, g_x^T) direction. We relay on the activity analysis approach introduced by Koopmans (1951) and discussed in Mas–Colell et al. (1995), that can be empirically implemented by way of Data Envelopment Analysis, DEA, techniques (Banker et al., 1984). This approach to economic and technical efficiency measurement approximates the true but unknown technology through piecewise linear combinations of the observed data, which constitute a multidimensional production frontier—see Cooper, Seiford and Tone (2000) or Ray (2004) for an introduction to DEA within a production theory context. The DEA piecewise linear approximation of the technology (1) is given by

$$\mathrm{T} = \left\{ (x,y) : \sum_{i=1}^{\mathrm{I}} \mathrm{z}_i \, \mathrm{x}_{in} \leq \mathrm{x}_n,\ n = 1,...,\mathrm{N};\ \sum_{i=1}^{\mathrm{I}} \mathrm{z}_i \, \mathrm{y}_{im} \geq \mathrm{y}_m,\ m = 1,...,\mathrm{M};\ \sum_{i=1}^{\mathrm{I}} \mathrm{z}_i \ = \ 1,\ z \in R_+^{\mathrm{I}},\ i = 1,...,\mathrm{I} \right\}, \quad (13)$$

where z is an intensity vector whose values determine the linear combinations or *facets* which define the production frontier. As (13) is formulated, this approximation exhibits variables returns to scale since the sum of the intensity variables z_i is equal to one.

Let us start operationally with the directional distance function of Chambers et al. (1996, 1998) as implemented by Färe and Grosskopf (2000b), but introducing the price normalization restriction: $(pg_y + wg_x) = 1$, so it yields foregone profit due to technical inefficiency. The program that allows calculation of the technical efficiency measure in a particular direction (g_x^T, g_y^T) satisfying the price constraint is:[5]

$$\mathrm{D_T}(x_{i'}, y_{i'}; g_x^T, g_y^T) = \max_{\beta, z_i} \beta \quad (14)$$

[5] We note that we could render (14) more flexible by letting it solve for the directional vector (g_x^T, g_y^T), thereby projecting the evaluated firm in the direction leading to the reference benchmark with the highest possible profit. In a sense, the directional vector would also be endogenous, subject only to its non negativity constraint—graphically, the firm is forced to move northwesterly without pre-assigning any direction.

$$\text{s.t.} \quad \sum_{i=1}^{I} z_i\, x_{in} \le x_{i'n} - \beta g_{x_n}^T,\ n = 1,...,N,$$

$$\sum_{i=1}^{I} z_i\, y_{im} \ge y_{i'm} + \beta g_{y_m}^T,\ m = 1,...,M,$$

$$\sum_{i=1}^{I} z_i = 1,$$

$$z \in R_+^I.$$

As remarked in the second section, the directional vector (g_x^T, g_y^T) may adopt many values as long as it satisfies the price normalization constraint. Here we consider a directional vector that places equal weight on outputs expansion and inputs reduction, *i.e.* $g_{x_n} = g_{y_m} = 1/(\Sigma_{m=1}^{M} p_m + \Sigma_{n=1}^{N} w_n)$.[6] As previously stated, the value returned by (14) allows us to determine whether the evaluated firm is technically efficient or not, which is the information we need to categorize profit inefficiency as either technical: $D_T(x_{i'}, y_{i'};\ g_x^T,\ g_y^T) > 0$, or allocative: $D_T(x_{i'}, y_{i'};\ g_x^T,\ g_y^T) = 0$.

Precisely, the program that allows calculation of the directional profit efficiency measure (8) for firm *i'* in monetary terms, thereby identifying the profit maximizing benchmark in the optimal (g_y^*, g_x^*) direction is:

$$\mathrm{D_T}(x_{i'}, y_{i'}; g_x^*, g_y^*) = \max_{\beta, z_i, g_x^*, g_y^*} \beta \tag{15}$$

$$\text{s.t.} \quad \sum_{i=1}^{I} z_i\, x_{in} \le x_{i'n} - \beta g_{x_n}^*,\ n = 1,...,N,$$

$$\sum_{i=1}^{I} z_i\, y_{im} \ge y_{i'm} + \beta g_{y_m}^*,\ m = 1,...,M,$$

$$\sum_{m=1}^{M} p_m g_{y_m}^* + \sum_{n=1}^{N} w_n g_{x_n}^* = 1,$$

$$\sum_{i=1}^{I} z_i = 1, \quad z \in R_+^I,$$

[6] In the two dimensional space as the one considered in the following example this choice implies the projection of the firm along the vector (-1, 1)—along the 45º bisecting line. This particular choice corresponds to the l_∞-Hölder distance function introduced by Briec (1999).

which departs from the previous one in two crucial aspects. First, the directional vector is not pre–assigned, and therefore (15) searches for it given the price normalization constraint. Secondly, the elements of the optimal directional vector could adopt any value, positive and negative, as long as it is not null. By solving in a single run this "all-in-one" simple program we gain information about firm i's profit inefficiency, its optimal benchmark, and the optimal course that it should follow when planning and adopting profit improving strategies.[7] Particularly, when $D_T(x_{i'}, y_{i'};\ g_x^*, g_y^*) > 0$, so the firm is profit inefficient, and in conjunction with the directional distance function we can determine whether the source of the inefficiency is technical or allocative.

5. An illustrating example

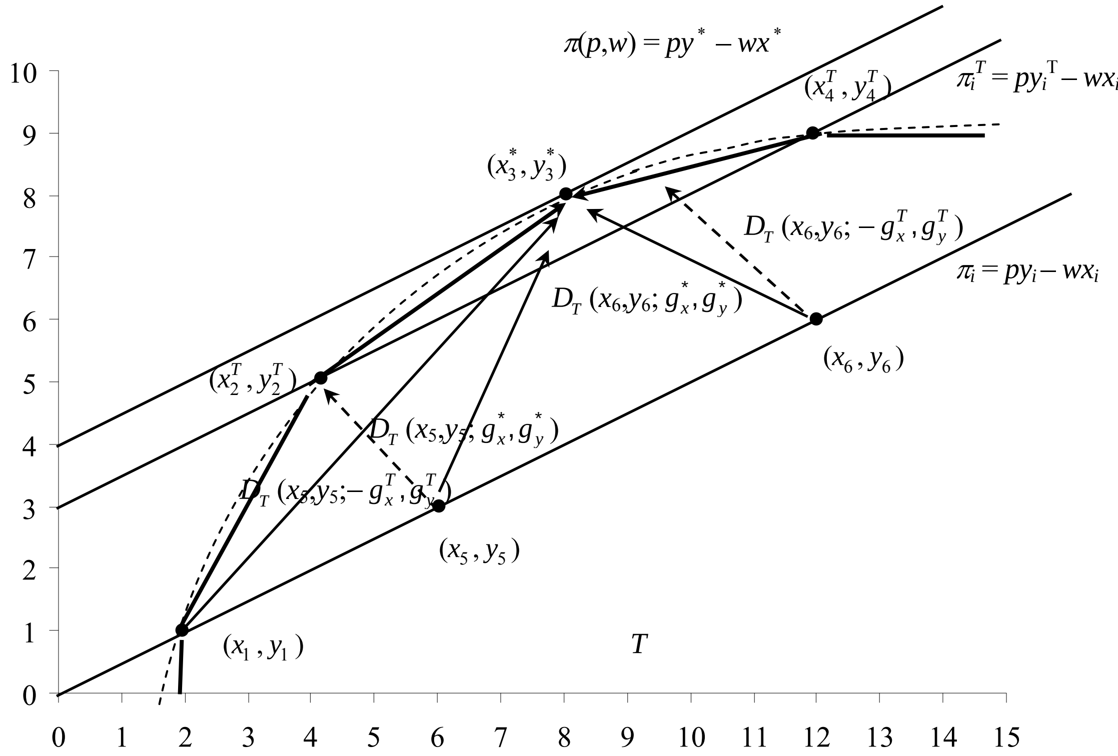
To illustrate the efficiency and productivity change model we consider a panel of six firms. Table 1 presents the different input—output vectors and their corresponding market prices. This example is also represented in Figure 2. An examination of the sample reveals that the third firm maximizes profit with $\pi(p,w) = py_3^* - wx_3^* =$ \$8, representing the economic maximizing benchmark for the remaining profit inefficient firms; particularly, the second and fourth firms present a profit inefficiency of six, e.g. OPE = $\pi(p,w) - (py_i - wx_i) = \pi(p,w) - \pi_2 =$ \$6, while for the first, fifth and sixth firms, whose profit is null, OPE = \$8.

Table 1. Example Data Set 1

Firm	x	y	
1	2	1	$p =$ 2
2	4	5	$w =$ 1
3	8	8	
4	12	9	
5	6	3	
6	12	6	

[7] Program (15) is clearly nonlinear. Nevertheless, it can be easily linearized by means of the following change of variables: $\gamma_{x_n} = \beta g_{x_n}^*$, $n = 1,...,N$, and $\gamma_{y_m} = \beta g_{y_m}^*$, $m = 1,...,M$. In this way, the constraint $\sum_{m=1}^{M} p_m g_{y_m}^* + \sum_{n=1}^{N} w_n g_{x_n}^* = 1$ is translated as $\sum_{m=1}^{M} p_m \gamma_{y_m} + \sum_{n=1}^{N} w_n \gamma_{x_n} = \beta$ and the objective function is not modified. Both programs are equivalent if and only if $\beta^* > 0$.

Figure 2. DEA Envelopment Technology (Data Set 1)



If we decompose profit inefficiency following the approach based on the directional distance function (2), with a choice of directional vector satisfying the price constraint while weighting inputs and output equally: $g_x = g_y = 1/(\Sigma_{m=1}^{M} p_m + \Sigma_{n=1}^{N} w_n) = 1/3$, so $(g_x^T, g_y^T) = (1/3,\ 1/3)$, [8] we learn that the first four firms are technically efficient, TE = 0, and therefore all profit inefficiency would be allocative—note that in Table 2 the signs of the directional vectors are consistent with the definitions (2) and (8) and their corresponding formulations (14) and (15), but reversed with respect to their graphical representation. On the contrary, for the fifth and sixth firms, their projections towards the frontier along this direction yield positive technically efficient values: $TE_5 = 6$ and $TE_6 = 7.2$, which equal foregone profit due to technical inefficiency. As a result, the residual between overall profit efficiency and technical efficiency is attributed to allocative (in)efficiency: $AE_5 = 2$ and $AE_6 = 1.8$.

When considering the directional profit efficiency measure yielding foregone profit due to technical or allocative inefficiency, along the optimal vector (g_y^*, g_x^*) projecting the evaluated firm

[8] In Figure 2 the directional distance function in the direction (g_x^T, g_y^T) = (−1/3, 1/3) is represented by the discontinuous arrows projecting the fifth and sixth firms to the envelopment frontier.

to its economic benchmark, the interpretation changes critically for technical inefficient firms. For technically efficient firms, as in the case of the standard directional distance function, the first, second and fourth firms fail to yield maximum profit due to allocative inefficiencies—since their technical efficiency value is null—and managers can be informed bout the necessary changes in their inputs and outputs mixes so as to reach full economic efficiency. For the first and second firms their optimal vectors are (–0.750, 0.875) and (–2.000, 1.500) respectively, implying that they should exploit the existing scale economies associated to more input usage and output increases, thereby attaining maximum profit. On the contrary, the directional vector (2.000, –0.500) associated to the fourth firm informs us that it endures scale diseconomies and, therefore, to reach maximum profit it should reduce its use of input thereby producing less output. As anticipated, for these firms defining the production frontier the decomposition of profit efficiency into its technical and allocative terms does not change from the existing approach based on the directional distance function, but it does for the fourth and fifth firms situating inside the technology.

Table 2. Overall Profit Efficiency Decomposition ($)

		D.D.F.					D.P.E.M			
Firm	OPE (8)	g_x^T	g_y^T	TE (2)	z	AE	g_x^*	g_y^*	TE	AE
1	8.000	0.333	0.333	0.000	$z_1 = 1$	8.000	–0.750	0.875	0.000	8.000
2	6.000	0.333	0.333	0.000	$z_2 = 1$	6.000	–2.000	1.500	0.000	6.000
3	0.000	0.333	0.333	0.000	$z_3 = 1$	0.000	-	-	0.000	0.000
4	6.000	0.333	0.333	0.000	$z_4 = 1$	6.000	2.000	–0.500	0.000	6.000
5	8.000	0.333	0.333	6.000	$z_3 = 1$	2.000	–0.250	0.625	8.000	0.000
6	8.000	0.333	0.333	7.200	$z_3=0.6$ $z_4=0.4$	1.800	0.500	0.250	8.000	0.000

For the fifth firm, reaching the optimal benchmark requires productive changes along the (–0.250, 0.625) optimal vector. As in the previous case of the first and fourth firm, this implies an input increase that is not compatible with the standard prescription that is obtained from the directional distance function model, even if the inputs and outputs adjustments proposed by both models results in profit increases. However, if we assume that there are adjustment costs proportional to input and output changes, it is clear that a monotone adjustment would be less costly—i.e. the shortest distance between the observed and optimal input—output bundles is a straight line. Since the fourth firm lies inside the production possibility set as signalled by the value of the directional distance function all profit inefficiency is due to technical reasons. The sixth firm illustrates an alternative case with a directional vector (0.500, 0.250) implying that to reach profit efficiency, this firm should behave as prescribed by the directional distance function model by reducing the input and increasing the output. However, we verify that in this latter case, any exogenous choice of the directional vector—as the neutral one considered in this example (−1/3, 1/3)—projects the evaluated firm to an allocative inefficient peer, unless it coincides precisely with the optimal direction given by the profit inefficiency measure (0.500, 0.250). This case shows that even when the optimal direction implies inputs and outputs adjustments compatible with those required with the directional distance function, the exogeneity of the directional vector translates into a subjective decomposition of profit inefficiency in technical and allocative components. In fact, given that the directional distance function for this firm is positive, we conclude that all inefficiency is technical, since the allocative residual is the result of the lack of flexibility of the directional distance function with respect to the orientation of the directional vector.

Finally, we deal with one situation that may occur when implementing the directional profit function by way of Data Envelopment Analysis techniques; that of multiple profit maximizing benchmarks.[9] In this situation the solution to program (15) would be compatible with a set of optimal directional vectors. This can be better illustrated resorting to Figure 3, depicting the DEA envelopment technology for the data set presented in Table 1 except for firm (x_3^*, y_3^*) that has been removed. For the same price vector $(p, w) = (2, 1)$, the profit maximizing benchmarks are now firms two and four, as well as the facet resulting from their lineal combination. In this situation, even if program (15) will yield the correct profit efficiency measure, nothing guarantees that the optimal directional vector (g_y^*, g_x^*) projects the evaluated firm to the closest benchmark, thereby implying the smallest input and output adjustments required to reach profit efficiency. However, it is possible to solve the following quadratic program minimizing the Euclidian distance between the evaluated firm and the closest benchmark maximizing profit, accounting for the price normalization constraint:

$$\min_{z, g_x^*, g_y^*} = \left[\pi(p, w) - \left(\sum_{m=1}^{M} p_m y_{i'm} - \sum_{n=1}^{N} w_n x_{i'n} \right) \right] \sqrt{\sum_{m=1}^{M} \left(g_{y_m}^* \right)^2 + \sum_{n=1}^{N} \left(g_{x_n}^* \right)^2} \tag{16}$$

[9] Mas-Collel et al. (1995:138) show that for the profit maximizing benchmark to be single valued, the technology must be strictly convex; a condition that it is not satisfied by the activity analysis approximation of the production technology, (13).

s.t.

$$\sum_{i=1}^{I} z_i x_{in} \leq x_{i'n} - \left[\pi(p,w) - \left(\sum_{m=1}^{M} p_m y_{i'm} - \sum_{n=1}^{N} w_n x_{i'n} \right) \right] g^*_{x_n}, n = 1,...,N,$$

$$\sum_{i=1}^{I} z_i y_{im} \geq y_{i'm} + \left[\pi(p,w) - \left(\sum_{m=1}^{M} p_m y_{i'm} - \sum_{n=1}^{N} w_n x_{i'n} \right) \right] g^*_{y_m}, m = 1,...,M,$$

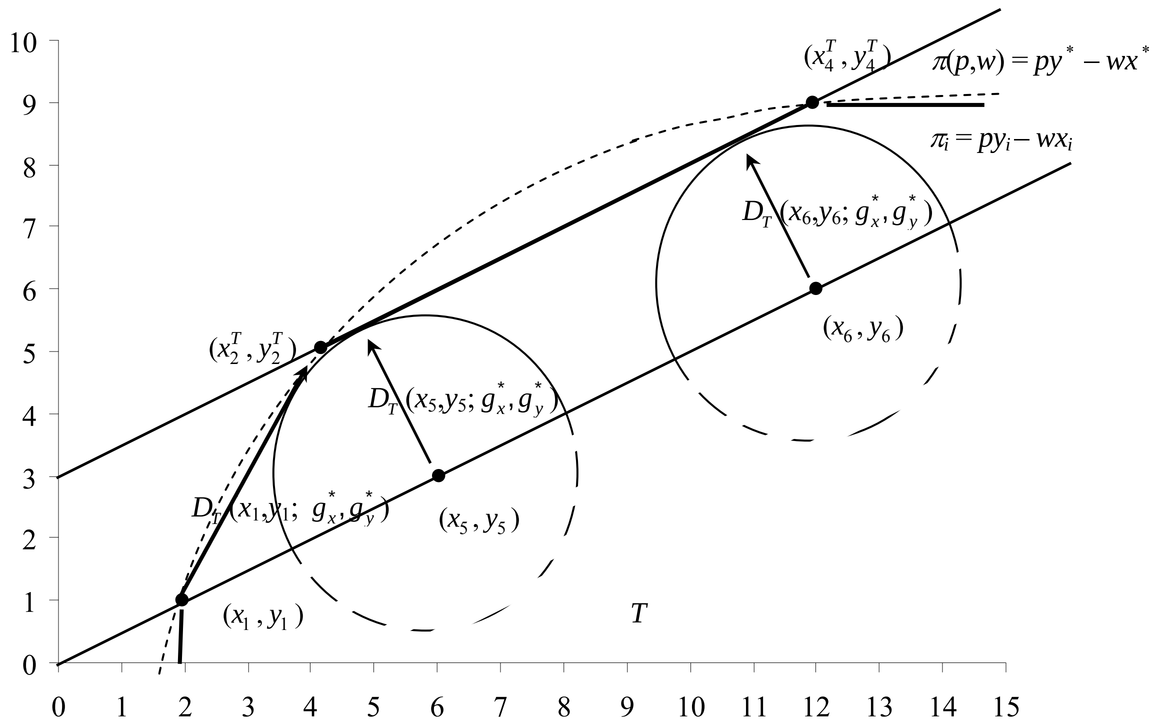
$$\sum_{m=1}^{M} p_m g^*_{y_m} + \sum_{n=1}^{N} w_n g^*_{x_n} = 1,$$

$$\sum_{i=1}^{I} z_i = 1,$$

$$z_i \geq 0, i = 1,...,I,$$

which yields the optimal direction: (g^*_y, g^*_x) = $(x_{i'} - x^*, y^* - y_{i'})$, thereby identifying the closest profit maximizing benchmark (x^*, y^*) to the evaluated firm $(x_{i'}, y_{i'})$. Finally, solving program (15) exogenously using this optimal vector returns the directional profit efficiency measure. For the first firm the closest benchmark maximizing profit is the second firm, and the optimal direction satisfying the price normalization constraint (g^*_y, g^*_x) = (0.333, 0.667), and overall profit efficiency is equal to six dollars. The results for the fifth and sixth firms are identical, presenting the same optimal directional vectors (–0.25, 0.5) and overall profit inefficiency.

Figure 3. DEA Envelopment Technology for Data Set 2—(x_3^*, y_3^*) removed.



6. Conclusions

The issue of decomposing profit inefficiency so as to identify its sources as technical and allocative have gained recent attention since the seminal papers of Shephard (1953) and Farrell (1957) respectively introducing the concept of distance function and its interpretation as a measure of technical efficiency. Until now the theoretical and empirical decomposition was based on the calculation of a distance function or technical efficiency measure that allows determination of the projected profit level, and, once compared to maximum profit, the difference would yield allocative efficiency as a residual. However, the calculation of technical efficiency assumes that inputs should be reduced while outputs should be increased. An assumption that derives from the initial partially oriented measures associated to the output or input distance functions, but that would not be longer necessary in flexible frameworks as the ones associated to the oriented directional distance function introduced by Chambers et al. (1996, 1998), the generalized distance function of Chavas and Cox (1999), or non-oriented additive approaches as in Cooper et al. (1999) or, more recently, Portela and Thanassoulis (2005, 2007) and Ruiz and Sirvent (2010). Nevertheless, these contributions are limited by restricting the adjustments of inputs and outputs in the way already mentioned and, for the literature dealing with particular orientations, by relying on an exogenously given directional vector.

In this article we dispose of this assumption by introducing the concept of the directional profit efficiency measure within the theoretical context of the directional distance function. This new measure renders the choice of the directional vector endogenous, and assuming a profit maximizing behavior, determines the difference between observed and maximal profit along an optimal path that projects any firm to its economic benchmark. From a theoretical—conceptual—perspective, and resorting to the directional distance function, we are able to categorize profit inefficiency as either technical (if the firm is technically inefficient lying inside the technology) or allocative (if the firm is technically efficient by defining the production frontier). In the light of this analytical proposal, and from a productivity analysis perspective, the standard decomposition of profit inefficiency in technical and allocative components, respectively associated to engineering practices and market behavior, seems artificial since it originates from the limiting assumption about inputs and outputs adjustments, and the subjective choice of a directional vector. In some way the debate on which way is better to achieve overall profit efficiency within a two-step procedure (first technical, then allocative or vice versa), particularly when it implies non monotone inputs and output adjustments is somehow forced, and driven by the theoretical modeling framework existing to date. Of course it can be rationalized in different ways, as Bogetoft et al.

(2006) do by calling upon the double loop learning process discussed by Agryris and Schön (1978).[10]

From a managerial and organizational perspective, and when used to prescribe profit efficiency strategies to managers, the standard setting struggles to justify two-step non monotone adjustments of inputs and outputs. Supposedly, managers should deal with either technical or allocative inefficiencies in the first place, thereby solving any engineering or market malpractices, and then deal with the remaining inefficiency source. However, when the first step requires inputs reduction, e.g. when solving engineering or technical inefficiencies, and the second one implies rehiring them so as to achieve market or allocative efficiency, it seems probable that the firm will incur in extra adjustments costs. A direct strategy resulting in monotone inputs and outputs and adjustment will be normally favored by managers, as it does not lead to organizational contradictions that may be confusing. This means that the simple one-step strategy represented by our model should be the norm, and that non-monotone two-step models the exception. In our model, this simplicity is identified with an optimal directional vector that projects the firm onto its profit maximizing benchmark, and results in a clear cut categorization of the inefficiency sources, either technical and allocative (and in the standard approach the decomposition depends on the exogenous and subjective choice of the directional vector). By doing so managers are informed about the necessary changes that they have to make so as to reach the desired economic goal without incurring in costly adjustments that are primarily driven by theoretical and modeling limitations rather than real life reasoning.

Finally, we render the directional profit efficiency measure operational by introducing the necessary optimizing programs that allow its calculation, and relate it to the existing directional distance function as implemented by Fare and Grosskopf (2000b). The DEA programs are simple to solve computationally and yield all the information required to take action on profit improving strategies. To show its feasibility we illustrate the proposed model with a simple exercise based on a sample data. By clearing the way for the empirical application of the directional profit efficiency measure, it can be used to study a number of issues based on an optimizing economic behavior, either profit, cost or revenue, and specific topics of efficiency and productivity measurement. For example, as in the decomposition of profit change indicators based on revenue and costs differences

[10] These authors state that "the natural choice is the change of the overall procedures −technical and allocative−(double loop learning), and then to perform those procedures in the best possible way (single loop learning)−refereed to sequential technical and allocative adjustments (or vice versa). We argue that in the new analytical framework we proposed there is no need to resort to single loop learning as implied by the theory, since managers can carry out organizational changes taken jointly into account technical and allocative criteria.

rather than ratios (termed Bennet-Bowley by Chambers (1998)) into productivity (Luenberger) and allocative components, as well as in environmental profit efficiency studies accounting for undesirable outputs as in Ball (1991). The range of potential applications related to the economic behavior if the firm is wide and open to substantial contributions both from the economics and OR/MS fields.

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